Recent Results in Analytic Number Theory

by

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Let us start with a Couple of Interesting Results Related to Riemann's Zeta Function:

1) Euler-type Formula
2) Explicit Product Representation of the Zeta Function.

- For Re $z > 1$, **Riemann zeta function** is defined by $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$.
- After analytic continuation using the key formula:

$$\zeta(z) = 2(2\pi)^{z-1} \sin\frac{\pi z}{2} \Gamma(1-z)\zeta(1-z), \operatorname{Re} z < 1$$

ζ is analytic everywhere except for a simple pole at z=1 with residue 1,

ζ vanishes at the negative even integers and those are called **trivial zeros.** Other zeros (plenty as we'll see) are called **non-trivial zeros** and they are INSIDE the **Critical Strip** $\{z : 0 \le \operatorname{Re} z \le 1\}$,

In 1859 Riemann conjectured that ALL non-trivial zeros of the zeta function have real part equal to ½ .

!) Finding the exact value that $\sum_{n=1}^{\infty} \frac{1}{n^3}$

converges to is one of the most notorious problems that did not even yield to Euler. A less difficult problem to consider is to find a

representation of $\zeta(3)$ in terms of $\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}$

only which would be a beautiful result similar to Euler's

$$\zeta(2) = 3 \sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}}$$

I have discovered the following representation (Maple was helpful)

$$\zeta(3) = -\frac{\sqrt{3}}{18}\pi^3 + \frac{3\sqrt{3}}{4}\pi\sum_{n=1}^{\infty}\frac{1}{(3n-2)^2} - \frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^3\binom{2n}{n}}$$

which I then proved mathematically.

However, the simply-looking series $\sum_{n=1}^{\infty} \frac{1}{(3n-2)^2} = \sum_{n=1}^{\infty} \frac{1}{(3n+1)^2}$

is hard to evaluate.

Possible strategy: Use a Computer Algebra System to find another (integer) relation

among $\zeta(3), \pi \sum_{n=1}^{\infty} \frac{1}{(3n-2)^2}, \sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}$

which when coupled with a mathematical proof yields an Euler-type representation for $\zeta(3)$.

The completed zeta function is defined by

$$\xi(z) = \frac{1}{2} z(z-1)\pi^{-\frac{z}{2}}\zeta(z)\Gamma(\frac{z}{2})$$

Some properties:

$$\xi(z) = \xi(1-z)$$

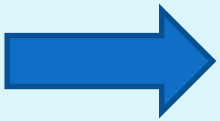
Entire
of order one and infinite type
(and so) has infinitely many zeros

RH can be stated as :

All zeros of ξ are on the critical line Re(z)=1/2

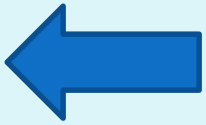
My plan:

Zeta function



Completed zeta function

Product representation of completed zeta function



Product representation of zeta function

Let me find a canonical representation of $\xi(z)$:

Since the zeta function has a simple pole at *z=1* with residue 1,

$$\xi(1) = \frac{1}{2}$$

Using the functional equation of the completed zeta function,

$$\xi(0) = \frac{1}{2}$$

Then

$$\xi(z) = e^{A} e^{Bz} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) \exp \frac{z}{z_n}$$

where $\{z_n\}$ are the non-zero zeros of the completed zeta function and hence, using the definition of the completed zeta function, are the non-trivial zeros of the zeta function and A and B are complex constants.

Now $\xi(0)=\frac{1}{2}$ implies that $e^{A}=\frac{1}{2}$ and so

$$\xi(z)=\frac{1}{2}e^{Bz}\prod_{n=1}^{\infty}(1-\frac{z}{z_n})\exp\frac{z}{z_n}$$

The effort to find B using $\xi(1)=\frac{1}{2}$ leads to

$$1=e^{B}\prod_{n=1}^{\infty}(1-\frac{1}{z_n})\exp\frac{1}{z_n}$$

Consider the product

$$p=\prod_{n=1}^{\infty}(1-\frac{1}{z_n})\exp\frac{1}{z_n}$$

Then

$$p^{z}=\prod_{n=1}^{\infty}(1-\frac{1}{z_n})^{z}\exp\frac{z}{z_n}$$

Therefore

$$\xi(z) = \frac{1}{2} e^{Bz} p^z \prod_{n=1}^{\infty} (1 - \frac{z}{z_n})(1 - \frac{1}{z_n})^{-z}$$

But $\xi(1) = \frac{1}{2}$ implies that $e^B p = 1$ and as a result

$$\xi(z) = \frac{1}{2} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n})(1 - \frac{1}{z_n})^{-z}$$

I have just shown that

$$\xi(z) = \frac{1}{2}\prod_{n=1}^{\infty}(1-\frac{z}{z_n})(1-\frac{1}{z_n})^{-z}$$

which together with

$$\frac{1}{\Gamma(z)} = z\prod_{n=1}^{\infty}(1+\frac{z}{n})(1+\frac{1}{n})^{-z}$$

yields the following interesting explicit representation:

$$\zeta(z) = \frac{1}{z-1}\ \pi^{\frac{z}{2}} \prod_{n=1}^{\infty} (1+\frac{z}{2n})\ (1-\frac{z}{z_n})\ (1+\frac{1}{n})^{-\frac{z}{2}} (1-\frac{1}{z_n})^{-z}$$

Pleasant Surprise:

Our discovered explicit representation above of the zeta function serves as a generalization of Wallis Formula:

Corollary:

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\cdots$$

Proof of corollary:

For $z \in C - \{0,1\}$ we can rewrite the representation as:

$$(z-1)\zeta(z) = \frac{\pi^{\frac{z}{2}}}{\frac{z}{2}} \underbrace{\prod_{n=1}^{\infty}(1-\frac{z}{z_n})(1-\frac{1}{z_n})^{-z}}_{2\xi(z)\,EntireHenceContinuous} \underbrace{\frac{z}{2}\prod_{n=1}^{\infty}(1+\frac{z}{2n})(1+\frac{1}{n})^{-\frac{z}{2}}}_{\frac{1}{\Gamma(\frac{z}{2})}\,EntireHenceContinuous}$$

The result now follows using $\lim_{z\to 1}(z-1)\zeta(z) = 1$.

3) **Goldbach Conjecture**

(in a letter to Euler in 1742):

Every natural number > 5 is a sum of 3 primes.

Euler believed it and phrased it as:

Every even number > 4 is a sum of 2 primes.

Nice Result:

(Ivan M.)Vinogradov Theorem (1937): Every sufficiently large ODD number $n>N$ is the sum of 3 primes:

Sufficiently large involves $N=3^{3^{15}} \cong 10^{70000}$ Brodozkin (Vinogradov student 1956), refined to $(2)10^{1346}$ by Liu and Wang (2002).
Still too large to be checked by computer.

Surprise!

- Every odd number > 5 can be written as the sum of 3 primes.

(Harald Helfgott at École normale supérieure and his May 13, 2013 133-page paper "Major arcs for Goldbach's Theorem" available on arxiv.org):

Summary: Helfgott success was the result of refining the N in Vinogradov Theorem to $(8.875)10^{30}$ (much smaller than $(2)10^{1346}$) and the help of Platt, Numerical Verification of the Ternary Goldbach Conjecture up to $(8.875)10^{30}$, Experimental Mathematics, Volume 22, Issue 4, 2013.

Prime numbers differing by 2 are called **twin primes.**

The 2300-year old (Euclid) twin prime conjecture states that the number of twin primes is infinite.

In the abstract of his talk “Revenge of the twin prime conjecture” at MathFest 2007, Daniel Goldston states that “two years ago Pintz,Yildirim, and I proved that there always exist primes that are very close together- very close meaning much closer than the average distance between neighboring primes. Our method also proves that if the primes are well distributed in arithmetic progressions, then one can obtain results not too far from the twin prime conjecture.

For example, **if the Elliott-Halberstam Conjecture is true then there are infinitely many pairs of primes with difference 16 or less.**

With these successes I was hopeful that before too long our method could be pushed to unconditionally show that there are infinitely often pairs of primes closer than some fixed bounded distance, i.e. bounded gaps, a giant step towards the twin prime conjecture. In this talk I will discuss the method and why perhaps further progress towards bounded gaps and the twin prime conjecture is going to be difficult, although I will be delighted to be proved wrong."

Another Surprise!

4) In his paper **Bounded gaps between primes**, Annals of Math, Volume 179, Issue 3, 2014, 1121-1174, **Yitang Zhang** proved that there is some natural number $N \leq 70{,}000{,}000$ such that **there are infinitely many pairs of primes that differ by *N***.

This is a spectacular result even though it does not prove that there are an infinite number of twin primes (where N would be 2).

Taking it further, the Polymath 8 project, started by Fields medalist Terence Tao at UCLA, has now, with the assistance of James Maynard, reduced the upper bound on the separation from 70,000,000 to 246.

Maynard and, independently, Polymath:
N=12 assuming the Elliott-Halberstam conjecture
(*roughly speaking* primes are strongly equidistributed in arithmetic progressions. After all, Tao and Green Theorem states that for every integer $k \geq 1$, the prime numbers contain an arithmetic progression of length k):

Technically, with φ *being Euler totient function* that counts the positive integers less than or equal to *n* that are relatively prime to *n*, *for all* $0 < \vartheta < 1$ *and all* $A>0$ *and all* $x>2$, there is $C>0$ s. t.

$$\sum_{1\le q\le x^{\vartheta}} \max_{(a,q)=1} \left|\pi(x;q,a)-\frac{\pi(x)}{\varphi(q)}\right| \le Cx\log^{-A} x.$$

To state the Generalized Elliott–Halberstam conjecture :

The Dirichlet convolution $\alpha \star \beta : \mathbb{N} \to \mathbb{C}$ of two arithmetic functions $\alpha, \beta : \mathbb{N} \to \mathbb{C}$ is defined as

$$\alpha \star \beta(n) = \sum_{d|n} \alpha(d)\beta\left(\frac{n}{d}\right)$$

For any function $\alpha: \mathbb{N} \to \mathbb{C}$ with finite support (that is, α is non-zero only on a finite set) and any primitive congruence class $a(q)$, we define the (signed) discrepancy $\Delta(\alpha; a(q))$ to be

$$\Delta(\alpha; a(q)) = \sum_{n=a(q)} \alpha(n) - \frac{1}{\varphi(q)} \sum_{(n,q)=1} \alpha(n)$$

Polymath :

N=6 assuming the Generalized Elliott-Halberstam conjecture

(Let $a(q)$ be the congruence class $\{a+nq : n \in Z\}$ and consider the primitive congruence classes

$$(Z/qZ)^{\times} := \{a(q) : (a,q) = 1\}$$

Let $D = x^{1-\varepsilon(x)}$ for some function ε(x) > 0 and

$$\frac{x}{D} \leq N, M \leq D$$

be such that NM ~ x, and let α, β be coefficient sequences at scale N,M. Then

$$\sum_{q \le D} \sup_{a \in (Z/qZ)^{\times}} |\Delta(\alpha, \beta; a(q))| \ll x \log^{-A} x$$

for any fixed $A \ge 1$)

(relevance to twin, cousin, and sexy primes).