Euler-type Formula

by

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- For Re *z>1*, the **Riemann zeta function** is defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

- After analytic continuation, ζ is analytic everywhere except for a simple pole at z=1 with residue 1.
- ζ vanishes at the negative even integers and those are called **trivial zeros.** Other zeros (plenty as we'll see) are called **non-trivial zeros** and they are INSIDE the **Critical Strip** $\{z : 0 \le \mathrm{Re}\, z \le 1\}$.
- In 1859 Riemann conjectured that ALL non-trivial zeros of his zeta function have real part equal to ½ .

Finding the exact value that $\sum_{n=1}^{\infty} \frac{1}{n^3}$

converges to is one of the most notorious problems that did not even yield to Euler. A less difficult problem to consider is to find a representation of $\zeta(3)$ in terms of $\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}$

only which would be a beautiful result similar to Euler's

$$\zeta(2) = 3 \sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}}$$

I have obtained the following representation

$$\zeta(3) = -\frac{\sqrt{3}}{18}\pi^3 + \frac{3\sqrt{3}\pi}{4}\sum_{n=1}^{\infty}\frac{1}{(3n-2)^2} - \frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^8\binom{2n}{n}}$$

(details consume time and are available via “Exploring New Identities…”)

The simply-looking series $\sum_{n=1}^{\infty} \frac{1}{(3n-2)^2}$ = $\sum_{0}^{\infty} \frac{1}{(3n+1)^2}$, however, is hard to evaluate.

The idea is to use AI as a tool to evaluate

$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)^2}$$

at the very best or at least express it in terms of $\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}$ or $\zeta(3)$ but not both.